New Classes of Off-Diagonal Cosmological Solutions in Einstein Gravity

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Abstract In this work, we apply the anholonomic deformation method for constructing new classes of anisotropic cosmological solutions in Einstein gravity and/or generalizations with nonholonomic variables. There are analyzed four types of, in general, inhomogeneous metrics, defined with respect to anholonomic frames and their main geometric properties. Such spacetimes contain as particular cases certain conformal and/or frame transforms of the well known Friedman-Robertson-Walker, Bianchi, Kasner and Gödel universes and define a great variety of cosmological models with generic off-diagonal metrics, local anisotropy and inhomogeneity. It is shown that certain nonholonomic gravitational configurations may mimic de Sitter like inflation scenarios and different anisotropic modifications without satisfying any classical false-vacuum equation of state. Finally, we speculate on perspectives when such off-diagonal solutions can be related to dark energy and dark matter problems in modern cosmology.

Keywords Anisotropic cosmology · Off-diagonal metrics · Exact solutions in gravity · Nonholonomic deformations

1 Introduction

Modifications of general relativity (GR) theory and new classes of cosmological solutions have received much attention as attempts to account for dark energy and recent observations from the Wilkinson Microwave Anisotropic Probe (WMAP), see [1–6] for reviews. There is certain evidence of relatively small anisotropic departures from the standard Friedmann-Robertson-Walker (FRW) model. However, we argue that it may be possible to involve more general classes of anisotropic and/or inhomogeneous cosmological solutions, described by

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generic off-diagonal metrics[1] in GR, in order to explain in the bulk the existing experimental data and examine various types of cosmological scenaria.

A series of Bianchi models with anisotropies has been analyzed. There is a classification of Bianchi metrics arranging all possible spatially homogeneous models depending on the symmetry properties of their spatial hypersurfaces, see recent developments and references in [7, 8]. A few of Bianchi universes contain the FRW model as a limiting case.[2] But even the best such models (the so-called Bianchi VII_h class) seem to be inconsistent [9] with WMAP data.

Another classes of anisotropic and/or inhomogeneous solutions are used in Kasner and Gödel cosmological models [10–16]. Nevertheless, the problem to construct the cosmological model and related solutions of gravitational and matter field equations which would describe most realistically the existing experimental data is still unsolved in modern gravity, cosmology and astrophysics.

Recently, we developed the so-called anholonomic deformation method of constructing exact solutions in gravity [17] (see examples and reviews in [18–22]). Perhaps, this is the unique existing at present geometric method providing a formalism for generating very general classes of solutions of gravitational field equations in GR and various high dimension/metric-affine, Lagrange-Finsler, noncommutative or other extensions. The method is based on nonlinear connection geometry originally elaborated for Finsler spaces, see details in [23] and, in relation to standard theories of physics, in [20, 22]. In this work, we shall not concern any questions related to details for scenaria of "Finsler cosmology" but consider some geometric techniques for generating cosmological solutions parametrized by generic off-diagonal metrics. The goal of this paper is to construct and analyze new classes of cosmological metrics for four dimensional, 4-d, pseudo-Riemannian spacetimes.

The off-diagonal anisotropic cosmological solutions to be considered in this work are for GR. We shall use an auxiliary linear connection (the so-called canonical distinguished connection, d-connection), for which the Einstein equations can be solved in general form with respect to some adapted nonholonomic frames of reference.[3] Such a connection is also metric compatible and completely defined by the metric structure but contains nontrivial torsion terms induced nonholonomically by some off-diagonal coefficients of metric. Having constructed generalized exact (cosmological) solutions for an auxiliary connection, we have to impose certain additional constraints on coefficients of metrics in order to "extract" exact solutions for the Levi-Civita connection.

We emphasize that the anholonomic deformation method allows us to generate "almost all" classes of solutions when the time like coordinate is contained as a "nonholonomic" one. Such generic off-diagonal solutions possess at least a Killing vector symmetry and, in general, depend additionally on two space coordinates. Various models of nonhomogeneous

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[1]Such metrics can not be diagonalized by any coordinate transforms.

[2]The paradigm of modern cosmology is based on the Friedmann-Robertson-Walker (FRW) metric which is derived as a spherical symmetric solution of the Einstein equations assuming homogeneity and isotropy on large scales.

[3]In modern literature on mathematics and physics, there are considered three equivalent terms: nonholonomic, anholonomic and/or non-integrable (for convenience, we shall use all such terms). Here, we also note that our approach should not be confused with the so-called Cartan's moving frame method when some geometric/physical objects are re-defined with respect to some more convenient frames of references/local basis. We consider nonholonomic deformations of geometric/physical objects (for instance, deformations of the Levi-Civita connection) by imposing non-integrable constraints on the dynamics of gravitational fields and anholonomic frames with associated nonlinear connections structure (the last one being defined as a conventional horizontal (h) and vertical (v) spacetime splitting).

and locally anisotropic cosmological models[4] can be elaborated. The Einstein equations are very complex systems of nonlinear partial differential equations. If we perform cosmological approximations for an explicit metric ansatz (for instance, considering only the dependence on time by averaging on space coordinates and imposing certain additional spacetime symmetries), we get some systems of nonlinear ordinary differential equations. Even we may be able to solve such systems in a general form, we get only a very restricted subclass of cosmological solutions. This way we "cut" the bulk of nonlinear gravitational interactions and loose a number of important off-diagonal solutions.

In our approach, we can integrate the Einstein equations in some very general forms. Performing approximations for general solutions (and not for certain coefficients of systems of nonlinear equations), we find new classes of cosmological metrics which extend the already known families of metrics for Bianchi universes, Kasner spacetimes etc. We note, that it is not possible to derive such cosmological metrics working directly with the Levi-Civita connection and local coordinate frames. The surprising property of separation of equations[5] exists for a more general type of connections which are also completely (and uniquely, but with a different geometric meaning) determined by the metric structure. At the first step, we can construct solutions for generalized connections and then, the second step, we have to constrain some coefficients of metrics in order to generate (in our case, cosmological) solutions in Einstein gravity. Solutions with "un-constrained" off-diagonal metrics also present a substantial interest for modern cosmology because they can be related to more general models of string/brane cosmology etc.

In our further works, we are going to provide an exhaustive study of generic off-diagonal and locally anisotropic metrics, and related cosmological models, in Einstein and Lagrange-Finsler theories of gravity that possess a FRW limit. The purpose of such constructions is to characterize as full as possible the cosmological spacetimes with anisotropies and generic off-diagonal nonlinear interactions and thus provide the strongest possible constraints on exotic cosmologies. In the present paper, however, we focus on the very specific question of whether general off-diagonal cosmological solutions can be constructed in Einstein gravity and if such models necessarily involve de Sitter stadia and possible inflation induced by nonlinear gravitational interactions.

This paper is organized as follows:

In Sect. 2, we present necessary geometric preliminaries on the nonlinear connection formalism and nonholonomic deformations of metrics, connections and frames in (pseudo) Riemannian spacetimes. We outline certain classes of important cosmological solutions which in this work will be deformed nonholonomically into generic off-diagonal solutions. The Einstein equations are equivalently formulated for two types of important linear connections. We use the fact that for the so-called canonical distinguished connection, the gravitational field equations can be separated with respect to adapted nonholonomic frames.

In Sect. 3, we prove that the Einstein equations can be integrated in very general forms containing all possible inhomogeneous and locally anisotropic cosmological solutions. There are analyzed some important parametrizations and subclasses of such off-diagonal solutions.

Section 4 is devoted to explicit constructions of generic off-diagonal cosmological solutions. We derive families of anisotropic spacetimes containing in certain limits the FRW,

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[4]We shall use also the equivalent terms 'inhomogeneous/nonhomogeneous' and say that the solutions are (locally) anisotropic if the geometric constructions are defined for off-diagonal metrics with dependencies both on time and space coordinates.

[5]It should be not confused with separation of variables.

Bianchi, Kasner and Gödel type configurations. We show that imposing nonintegrable (non-holonomic) constraints on the nonlinear dynamics of off-diagonal gravitational interactions we can model various types of anisotropic and de Sitter solutions.

We summarize and discuss the results in Sect. 5.

2 Nonholonomic Deformations of Cosmological Solutions

In this section, we give general features of the geometry of nonholonomic deformations and apply this formalism for constructing exact off-diagonal cosmological solutions. We follow the notations of [19–22] were details and references can be found.

2.1 Geometric Preliminaries

Let us consider a (pseudo) Riemannian 4-d manifold $\mathbf{V}$ endowed with a metric $\mathbf{g} = g_{\alpha\beta}(u^\gamma)du^\alpha \otimes du^\beta$ of signature $(+, +, -, +)$ when local coordinates are parametrized in the form $u^\alpha = (x^i, y^a)$, where $x^i = (x^1, x^2)$ and $y^a = (y^3 = t, y^4 = y)$.[6] Indices $i, j, k, \ldots = 1, 2$ and $a, b, c, \ldots = 3, 4$ are used for a conventional $(2+2)$-splitting of dimension and general abstract/coordinate indices when $\alpha, \beta, \ldots$ run values 1, 2, 3, 4.

We denote by $\nabla = \{\Gamma^\alpha_{\beta\gamma}\}$ the Levi-Civita connection[7] with coefficients stated with respect to an arbitrary local frame basis $e_\alpha = (e_i, e_a)$ and its dual basis $e^\beta = (e^j, e^b)$. Contracting the first and third coefficients of the Riemannian curvature tensor $\mathcal{R} = \{R^\alpha_{\beta\gamma\delta}\}$ of ∇, we define the Ricci tensor, $\mathcal{R}ic = \{R_{\beta\delta} \doteq R^\alpha_{\beta\alpha\delta}\}$, which (in its turn) can be used for computing the scalar curvature $R \doteq g^{\beta\delta}R_{\beta\delta}$, where $g^{\beta\delta}$ is inverse to $g_{\alpha\beta}$. The Einstein equations on $\mathbf{V}$, for an energy-momentum source of matter $T_{\alpha\beta}$, are written in the form

$$R_{\beta\delta} - \frac{1}{2}g_{\beta\delta}R = \varkappa T_{\beta\delta}, \tag{1}$$

where $\varkappa = const$.

Our goal is to construct exact solutions of gravitational field equations (1) parametrized in the form

$$
\begin{aligned}
{}^\eta\mathbf{g} &= \eta_i(x^k, t) \, {}^\circ g_i(x^k, t)dx^i \otimes dx^i + \eta_a(x^k, t) \, {}^\circ h_a(x^k, t)\mathbf{e}^a \otimes \mathbf{e}^a, \\
\mathbf{e}^3 &= dt + \eta_i^3(x^k, t) \, {}^\circ w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + \eta_i^4(x^k, t) \, {}^\circ n_i(x^k, t)dx^i,
\end{aligned} \tag{2}
$$

for certain classes of coefficients (functions) to be defined below. In brief, we shall write such metrics as

$$\mathbf{g} = g_{ij}dx^i \otimes dx^j + h_{ab}(dy^a + N_k^a dx^k) \otimes (dy^b + N_k^b dx^k), \tag{3}$$

where, for (2), $g_{ij} = \text{diag}[g_i = \eta_i \, {}^\circ g_i]$ and $h_{ab} = \text{diag}[h_a = \eta_a \, {}^\circ h_a]$ and $N_k^3 = w_i = \eta_i^3 \, {}^\circ w_i$ and $N_k^4 = n_i = \eta_i^4 \, {}^\circ n_i$. The gravitational 'polarizations' η_α and η_i^a determine nonholonomic

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[6]In our works, we use conventions from [20, 22] when left up/low indices are used as labels for spaces and geometric objects. We state that $y^3 = t$ because such a parametrization will allow us to construct and write down the formulas for equations and solutions in a "most" simplified form.

[7]Which is uniquely defined by a given tensor $\mathbf{g}$ to be metric compatible, $\nabla\mathbf{g} = 0$, and with zero torsion; we follow the conventions established in our previous works [20, 22], including summarizing on "up-low" repeating indices if the contrary is not stated.

deformations of metrics, $^\circ \mathbf{g} = [\,^\circ g_i, \,^\circ h_a, \,^\circ N_k^a\,] \to^\eta \mathbf{g} = [\,g_i, h_a, N_k^a\,]$. Such transforms (with deformations of the frame, metric, connections and other fundamental geometric structures) are more general than those considered for the Cartan's moving frame method, when the geometric objects are re-defined equivalently with respect to necessary systems of reference.

Any set of coefficients N_k^a in (3) state on $\mathbf{V}$ some N-adapted frame, $\mathbf{e}_\alpha$, and dual frame, $\mathbf{e}^\beta$, structures (i.e. N-elongated partial derivatives, respectively, differentials)

$$\mathbf{e}_\alpha \doteq \left(\mathbf{e}_i = \partial_i - N_i^a \partial_a, \ e_b = \partial_b = \frac{\partial}{\partial y^b} \right), \tag{4}$$

$$\mathbf{e}^\beta \doteq \left(e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i \right). \tag{5}$$

Such local bases satisfy some nonholonomic relations

$$\left[\mathbf{e}_\alpha, \mathbf{e}_\beta \right] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = \mathbf{w}^\gamma_{\ \alpha\beta}(u)\,\mathbf{e}_\gamma, \tag{6}$$

with nontrivial anholonomy coefficients $\mathbf{w}^\alpha_{\ \beta\gamma}(u)$,

$$\mathbf{w}^a_{\ ji} = -\mathbf{w}^a_{\ ij} = \Omega_{ij}^a = \mathbf{e}_j N_i^a - \mathbf{e}_i N_j^a, \qquad \mathbf{w}^b_{\ ia} = -\mathbf{w}^b_{\ ai} = \partial_a N_i^b. \tag{7}$$

We can fix on $\mathbf{V}$ such systems of N-elongated frames when the sets of coefficients $\mathbf{N} = \{N_k^a\}$ define a Whitney splitting (in general, non-integrable) of tangent space $T\mathbf{V}$ to $\mathbf{V}$,

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \tag{8}$$

into conventional horizontal (h) and vertical (v) subspaces, respectively, $h\mathbf{V}$ and $v\mathbf{V}$. Such a geometric object defines a nonlinear connection (N-connection) structure.

For any metric $\mathbf{g}$ (3) on a spacetime $\mathbf{V}$, there is an infinite number of metric compatible linear connections D, satisfying the conditions $D\mathbf{g} = 0$, and completely defined by $\mathbf{g}$. A subclass of such linear connections can be adapted to a chosen N-connection structure $\mathbf{N}$, when the splitting (8) is preserved under parallelism, and called distinguished connections (in brief, d-connections). A general d-connection is denoted by a boldface symbol $\mathbf{D} = (hD, vD)$ distinguished into, respectively, h- and v-covariant derivatives, hD and vD. To construct exact solutions in gravity theories is convenient to work with the so-called canonical d-connection, $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^\gamma_{\ \alpha\beta}\}$, which with respect to N-adapted bases (4) and (5) is given by coefficients $\widehat{\mathbf{\Gamma}}^\gamma_{\ \alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$, for $h\widehat{D} = \{\widehat{L}^i_{jk}, \widehat{L}^a_{bk}\}$ and $v\widehat{D} = \{\widehat{C}^i_{jc}, \widehat{C}^a_{bc}\}$, where

$$\widehat{L}^i_{jk} = \frac{1}{2} g^{ir} \left(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk} \right),$$

$$\widehat{L}^a_{bk} = e_b(N_k^a) + \frac{1}{2} h^{ac} \left(\mathbf{e}_k h_{bc} - h_{dc} \, e_b N_k^d - h_{db} \, e_c N_k^d \right), \tag{9}$$

$$\widehat{C}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}, \qquad \widehat{C}^a_{bc} = \frac{1}{2} h^{ad} \left(e_c h_{bd} + e_c h_{cd} - e_d h_{bc} \right).$$

This canonical d-connection $\widehat{\mathbf{D}}$ and its torsion $\mathcal{T} = \{\widehat{\mathbf{T}}^\gamma_{\ \alpha\beta} \equiv \widehat{\mathbf{\Gamma}}^\gamma_{\ \alpha\beta} - \widehat{\mathbf{\Gamma}}^\gamma_{\ \beta\alpha}; \widehat{T}^i_{jk}, \widehat{T}^i_{ja}, \widehat{T}^a_{ji}, \widehat{T}^a_{bi}, \widehat{T}^a_{bc}\}$, where the nontrivial coefficients

$$\widehat{T}^i_{jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}, \qquad \widehat{T}^i_{ja} = \widehat{C}^i_{jb}, \qquad \widehat{T}^a_{ji} = -\Omega^a_{ji},$$
$$\widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N_j^c), \qquad \widehat{T}^a_{bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb}, \tag{10}$$

are completely defined by the coefficients of metric $\mathbf{g}$ (3) following the conditions that $\widehat{\mathbf{D}}\mathbf{g} = 0$ and the "pure" horizontal and vertical torsion coefficients are zero, i.e. $\widehat{T}^i_{jk} = 0$ and $\widehat{T}^a_{bc} = 0$.

Any geometric construction for the canonical d-connection $\widehat{\mathbf{D}}$ can be re-defined equivalently into a similar one with the Levi-Civita connection following formula

$$\Gamma^\gamma_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta}, \tag{11}$$

where the distortion tensor $Z^\gamma_{\alpha\beta}$ is constructed in a unique form from the coefficients of a metric $\mathbf{g}_{\alpha\beta}$,

$$Z^a_{jk} = -\widehat{C}^i_{jb} g_{ik} h^{ab} - \frac{1}{2}\Omega^a_{jk}, \qquad Z^i_{bk} = \frac{1}{2}\Omega^c_{jk} h_{cb} g^{ji} - \Xi^{ih}_{jk}\widehat{C}^j_{hb},$$

$$Z^a_{bk} = {}^+\Xi^{ab}_{cd}\widehat{T}^c_{kb}, \qquad Z^i_{kb} = \frac{1}{2}\Omega^a_{jk} h_{cb} g^{ji} + \Xi^{ih}_{jk}\widehat{C}^j_{hb}, \qquad Z^i_{jk} = 0, \tag{12}$$

$$Z^a_{jb} = -{}^-\Xi^{ad}_{cb}\widehat{T}^c_{jd}, \qquad Z^a_{bc} = 0, \qquad Z^i_{ab} = -\frac{g^{ij}}{2}\left[\widehat{T}^c_{ja} h_{cb} + \widehat{T}^c_{jb} h_{ca}\right],$$

for $\Xi^{ih}_{jk} = \frac{1}{2}(\delta^i_j \delta^h_k - g_{jk} g^{ih})$ and ${}^\pm\Xi^{ab}_{cd} = \frac{1}{2}(\delta^a_c \delta^b_d + h_{cd} h^{ab})$.

2.2 Limits to Known Cosmological Solutions

In this work, we shall construct new classes of cosmological solutions[8] with metrics ${}^n\mathbf{g}$, i.e. 'target' metrics, possessing certain limits, for $\eta_\alpha, \eta^a_i \to 1$, or $\eta^a_i \to 0$, to ${}^\circ\mathbf{g}$ (a 'prime' metric) which is a conformal, or frame/coordinate, transform of a well known FRW, Bianchi, Kasner, or another type solution, see reviews of results in [7, 8, 24, 25].

2.2.1 FRW Metrics

The FRW cosmological solution can be written in the form

$$_F\mathbf{g} = a^2(t)\left(\frac{dr \otimes dr}{1 - \kappa r^2} + r^2 d\theta \otimes d\theta\right) - dt \otimes dt + a^2(t) r^2 \sin^2\theta d\varphi \otimes d\varphi, \tag{13}$$

with $\kappa = \pm 1, 0$, when the coordinates and coefficients of metric are parametrized, respectively, in the form $x^1 = r, x^2 = \theta, y^3 = t, y^4 = \varphi$ (for spherical coordinates) and $_F g_1 = a^2/(1 - \kappa r^2)$, $_F g_2 = a^2 r^2/(1 - \kappa r^2)$, $_F h_3 = -1$, $_F h_4 = a^2(t) r^2 \sin^2\theta$ and $_F N^a_i = 0$.[9] This metric is an exact solution of equations (1) with a perfect fluid stress-energy tensor,

$$T^\alpha{}_\beta = \text{diag}[-p, -p, \rho, -p], \tag{14}$$

where ρ and p are the proper energy density and pressure in the fluid rest frame. The Einstein equations for ansatz (13) take the form of two coupled nonlinear ordinary differential equations (also called the Friedmann equations)

$$H^2 \equiv \left(\frac{a^*}{a}\right)^2 = \frac{1}{3}\rho - \frac{\kappa}{a^2} \tag{15}$$

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[8]In general, such spacetimes inhomogeneous and anisotropic.

[9]Instead of FRW as a 'prime' metric ${}^\circ\mathbf{g}$, we can consider any Bianchi, Kasner etc. cosmological solutions outlined in [4–6, 8, 9] and references therein.

and

$$H^* + H^2 = \frac{a^{**}}{a} = -\frac{1}{6}(\rho + 3p), \tag{16}$$

where the strong energy conditions for matter, $\rho + 3p \geq 0$, must be satisfied for an expanding universe.[10] The Hubble parameter $H \equiv \frac{a^*}{a}$ has the unit of inverse time and is positive (negative) for an expanding (collapsing) universe. The equations (15) and (16) may be combined (which is also related to the condition $\nabla_\alpha T^\alpha_{\ \beta} = 0$)

$$\rho^* + 3H(\rho + p) = 0.$$

For simplicity, we can consider $\kappa = 0$ with

$$_F\mathbf{g} = a^2(t)\,(dx \otimes dx + dz \otimes dz) - dt \otimes dt + a^2(t)dy \otimes dy, \tag{17}$$

for Cartezian coordinates and coefficients of metric parametrized, respectively, in the form $x^1 = x, x^2 = z, y^3 = t, y^4 = y$ and $^\circ g_1 = {}_F g_2 = a^2$, $_F h_3 = -1$ and $_F h_4 = a^2$ (in this case, the nontrivial coefficients of metric depend only on time like coordinate, t, but not on space like ones).

2.2.2 Bianchi Type Metrics

All possible spatially homogeneous but anisotropic relativistic cosmological models are arranged following the Bianchi classification corresponding to symmetry properties of their spatial hypersurfaces [8, 26, 27]. Such cosmological metrics can be parametrized by ortho-normal tetrad (vierbein) bases $e_{\alpha''} = e^{\alpha}_{\ \alpha''}\partial/\partial u^\alpha$, when

$$_B\mathbf{g}_{\alpha''\beta''} = {}_Be^{\alpha}_{\ \alpha''}\,{}_Be^{\beta}_{\ \beta''}\,{}_Bg_{\underline{\alpha\beta}} = \text{diag}[1, 1, -1, 1] \tag{18}$$

and

$$\left[{}_Be_{\alpha''},\,{}_Be_{\beta''}\right] = {}_Bw^{\gamma''}_{\ \alpha''\beta''}(t)\,{}_Be_{\gamma''},$$

with time dependent 'structure constants'

$$_Bw^{\gamma''}_{\ \alpha''\beta''}(t) = \epsilon_{\alpha''\beta''\tau''}n^{\tau''\gamma''}(t) + \delta^{\gamma''}_{\beta''}b_{\alpha''}(t) - \delta^{\gamma''}_{\alpha''}b_{\beta''}(t) \tag{19}$$

(determined by some diagonal tensor, $n^{\tau''\gamma''}$, and vector, $b_{\alpha''}$, fields) used for the mentioned classification. Depending on parametrization of such tensor and vector objects, there are constructed the so-called Bianchi universes which are closed, or not, to the homogeneous and isotropic FRW case. With nontrivial limits to observable cosmology, there are the so-called Bianchi I, V, VII_0, VII_h and IX cosmologies.

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[10]For our purposes, and following our former works on geometric methods in gravity and exact solutions [17, 19, 20], it is convenient to introduce a system of notations which is different form those used in standard books on cosmology (see, for instance, [1–6, 24, 25], were readers may consult details and references on modern cosmology).

2.2.3 Kasner Type Metrics

For instance, in four dimensional gravity, such a metric is written in the form

$$_K\mathbf{g} = t^{2p_1}dx \otimes dx + t^{2p_3}dz \otimes dz - dt \otimes dt + t^{2p_2}dy \otimes dy, \tag{20}$$

with $_Kg_1 = t^{2p_1}$, $_Kg_2 = t^{2p_3}$, $_Kh_3 = -1$, $_Kh_4 = t^{2p_2}$, $_KN_i^a = 0$, see details (including modern brane generalizations) and references in [10–12]. The constants p_1, p_2, p_3 define solutions of the Einstein equations if there are satisfied the conditions

$$2\,^3P = \,^2P - \,^1P, \tag{21}$$

for $(^1P)^2 = (p_1)^2 + (p_2)^2 + (p_3)^2$, $^2P = p_1 + p_2 + p_3$, $^3P = p_1 p_2 + p_2 p_3 + p_1 p_3$. Following the anholonomic deformation method, we shall generalize such solutions to generic off-diagonal cosmological configurations, see Sect. 4.1.

2.2.4 Gödel Model

The theoretical model for the study of rotating cosmology is determined by the Gödel solution [13]

$$_G\mathbf{g} = \,_Ga^2\left[dx \otimes dx + \frac{e^{2x}}{2}dz \otimes dz - (dt - e^x dz) \otimes (dt - e^x dz) + dy \otimes dy\right], \tag{22}$$

when $_Gg_1 =_G a^2$, $_Gg_2 = \frac{e^{2x}}{2}\,_Ga^2$, $_Gh_3 = -_Ga^2$, $_Gh_4 =_G a^2$, $_GN_i^a =_G w_i = -e^x$, $_GN_i^a = 0$. The matter is described as a dust with the energy density $_G\varepsilon$ and there is a nontrivial negative (with opposite sign to that introduced by Einstein) cosmological constant $_G\lambda$ determining the angular velocity $_G\omega$ of the cosmic rotation, $_G\omega^2 = 1/2_Ga^2 = 4\pi G_G\varepsilon = -_G\lambda$, with G as Newton's gravitational constant. The parametrizations of coordinates and coefficients of metrics are different from that considered in the former works, see a comprehensive review of results and references in [14–16]. In Sect. 4.1, we shall provide an off-diagonal generalization defining rotating universes with polarized cosmological constants and nonholonomic/constrained rotations in nontrivial backgrounds.

2.3 The Einstein Equations for Connections $\widehat{\mathbf{D}}$ and ∇

The Einstein equations (1) for a metric $\mathbf{g}_{\beta\delta}$ can be rewritten equivalently using the canonical d-connection $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\,^sR = \mathbf{\Upsilon}_{\beta\delta}, \tag{23}$$

$$\widehat{L}_{aj}^c = e_a(N_j^c), \qquad \widehat{C}_{jb}^i = 0, \qquad \Omega_{ji}^a = 0, \tag{24}$$

where $\widehat{\mathbf{R}}_{\beta\delta}$ is the Ricci tensor for $\widehat{\mathbf{\Gamma}}_{\alpha\beta}^\gamma$, $^sR = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\beta\delta}$ and $\mathbf{\Upsilon}_{\beta\delta}$ is such way constructed that $\mathbf{\Upsilon}_{\beta\delta} \to \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$. We emphasize here that if the constraints (24) are satisfied the tensors $\widehat{\mathbf{T}}_{\alpha\beta}^\gamma(10)$ and $Z_{\alpha\beta}^\gamma(12)$ are zero. This states that $\widehat{\mathbf{\Gamma}}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma$, with respect to N-adapted frames (4) and (5), see (11), even $\widehat{\mathbf{D}} \neq \nabla$.

In a series of our works [17, 19–22], we provided detailed proofs that for constructing exact solutions with generic off-diagonal metrics it is more convenient to work with the

canonical d-connection $\widehat{\mathbf{D}}$ than with the Levi-Civita connection ∇; the last one is considered to be the standard one for general relativity. The surprising thing is that the "nonholonomic" gravitational field equations (23) split in such a form, with respect to N-adapted frames (4) and (5), that the resulting system of partial differential equations (see below, for instance, (26)–(29)) can be solved in very general forms. In order to generate exact solutions for ∇, we have to impose additional constraints (24) on coefficients of metric $\mathbf{g}$ (3) (for instance, on $^{\eta}\mathbf{g}$ (2) if we wont to generate new classes of cosmological solutions).[11]

For an ansatz of type (2), the Einstein equations (23) for $\widehat{\mathbf{D}}$ with a general source of type[12]

$$\Upsilon^{\alpha}{}_{\beta} = \mathrm{diag}[\Upsilon_{\gamma}; \Upsilon_1 = \Upsilon_2 = \Upsilon_2(x^k, t); \Upsilon_3 = \Upsilon_4 = \Upsilon_4(x^k)] \qquad (25)$$

transform into a system of nonlinear partial differential equations with separation of equations for h- and v-components of metric and N-connection coefficients,

$$\widehat{R}_1^1 = \widehat{R}_2^2$$
$$= -\frac{1}{2g_1 g_2}\left[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1} \right] = -\Upsilon_4(x^k), \qquad (26)$$

$$\widehat{R}_3^3 = \widehat{R}_4^4 = -\frac{1}{2h_3 h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3} \right] = -\Upsilon_2(x^k, t), \qquad (27)$$

$$\widehat{R}_{3k} = \frac{w_k}{2h_4}\left[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3} \right]$$
$$+ \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4} \right) - \frac{\partial_k h_4^*}{2h_4} = 0, \qquad (28)$$

$$\widehat{R}_{4k} = \frac{h_4}{2h_3}n_k^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^* \right)\frac{n_k^*}{2h_3} = 0. \qquad (29)$$

In brief, we wrote the partial derivatives in the form $a^{\bullet} = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial t$. The ansatz (2) does not depend on variable y^4 (that why we do not have terms with $\partial/\partial y^4$).

The above system of equations can be solved in very general forms for arbitrary dimensions and signatures as we proved in [17, 19–22] (see also next section). In this work, we analyze "cosmological configurations" for $\widehat{\mathbf{D}}$ when $y^3 = t$ for generic off-diagonal metrics of type (2). New classes of cosmological conditions in general relativity, with the Levi-Civita connection ∇, will be extracted by imposing additional constraints

$$w_i^* = \mathbf{e}_i \ln|h_4|, \qquad \mathbf{e}_k w_i = \mathbf{e}_i w_k, \qquad n_i^* = 0, \qquad \partial_i n_k = \partial_k n_i \qquad (30)$$

satisfying the conditions (24).

---

[11]We note that it is not possible to solve the Einstein equations in general form working directly with ∇ because this way we do not get a separation of nonlinear partial differential equations. Our idea is to use a more general connection (also defined completely by the same metric), when the system of nonlinear equations can be integrated in some general forms, and than to constrain the solutions to generate metrics for the general relativity theory.

[12]Such parametrizations of energy-momentum tensors are possible by corresponding nonholonomic frame and/or coordinate frame transform for very general matter sources, including some important cases with cosmological constants and various models of locally anisotropic fluid/scalar field/spinor/gauge fields interactions on curved spaces.

3 General Cosmological Off-Diagonal Solutions

In this section, we construct in explicit form and analyze the properties of possible classes of solutions depending on time like variable t and on "horizontal" spacelike coordinates x^i of gravitational field equations (26)–(29), for the canonical d-connection, and of constraints (30) selecting Levi-Civita configurations.

3.1 Type 1: Solutions with $h^*_{3,4} \neq 0$ and $\Upsilon_{2,4} \neq 0$

Such metrics are defined by a metric ansatz

$$
\begin{aligned}
{}^{\eta}\mathbf{g} &= e^{\psi(x^k)}dx^i \otimes dx^i + h_3(x^k,t)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k,t)\mathbf{e}^4 \otimes \mathbf{e}^4, \\
\mathbf{e}^3 &= dt + w_i(x^k,t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_i(x^k,t)dx^i
\end{aligned}
\tag{31}
$$

with the coefficients being solutions of the system[13]

$$
\ddot{\psi} + \psi'' = 2\Upsilon_4(x^k),
\tag{32}
$$

$$
h^*_4 = 2h_3 h_4 \Upsilon_2(x^i,t)/\phi^*,
\tag{33}
$$

$$
\beta w_i + \alpha_i = 0,
\tag{34}
$$

$$
n^{**}_i + \gamma n^*_i = 0,
\tag{35}
$$

where

$$
\phi = \ln\left|\frac{h^*_4}{\sqrt{|h_3 h_4|}}\right|, \qquad \alpha_i = h^*_4 \partial_i \phi, \qquad \beta = h^*_4 \phi^*, \qquad \gamma = \left(\ln|h_4|^{3/2}/|h_3|\right)^*.
\tag{36}
$$

For $h^*_4 \neq 0$; $\Upsilon_2 \neq 0$, we get $\phi^* \neq 0$. Prescribing any nonconstant $\phi = \phi(x^i,t)$ as a generating function, we can construct exact solutions of (32)–(35) solving respectively the two dimensional Laplace equation, for $g_1 = g_2 = e^{\psi(x^k)}$; integrating on t, in order to determine h_3, h_4 and n_i, and solving algebraic equations, for w_i. We obtain (computing consequently for a chosen $\phi(x^k,t)$)

$$
\begin{aligned}
g_1 = g_2 &= e^{\psi(x^k)}, \qquad h_3 = \pm \frac{|\phi^*(x^i,t)|}{\Upsilon_2}, \\
h_4 &= {}^0h_4(x^k) \pm 2\int \frac{(\exp[2\,\phi(x^k,t)])^*}{\Upsilon_2}dt, \\
w_i &= -\partial_i\phi/\phi^*, \qquad n_i = {}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right)\int [h_3/(\sqrt{|h_4|})^3]dt,
\end{aligned}
\tag{37}
$$

where ${}^0h_4(x^k)$, ${}^1n_k(x^i)$ and ${}^2n_k(x^i)$ are integration functions. We have to fix a corresponding sign $\pm$ in order to generate a necessary local signature of type $(+ + -+)$ for some chosen ϕ, Υ_2 and Υ_4.

Here we note that the general off-diagonal solutions (13) include as particular cases the solutions for a nontrivial cosmological constant $\Upsilon_i = \lambda$, or nonholonomic configurations with polarizations of such constants, $\lambda \to {}^h\lambda(x^k) = \Upsilon_4(x^k)$ and $\lambda \to {}^v\lambda(x^k,t) = \Upsilon_2(x^k,t)$.

---

[13] It is an equivalent of (26)–(29) for $h^*_{3,4} \neq 0$.

In order to construct exact solutions for the Levi-Civita connection, we have to constrain the coefficients (37) of metric (31) to satisfy the conditions (30). Such constraints restrict the class of generating and integration functions. For instance, we can chose that $^2n_k(x^i) = 0$ and $^1n_k(x^i)$ are any functions satisfying the conditions $\partial_i{}^1n_k = \partial_k{}^1n_i$. The constraints for $\phi(x^k, t)$ follow from constraints on N-connection coefficients $w_i = -\partial_i\phi/\phi^*$,

$$(w_i[\phi])^* + w_i[\phi]\,(h_4[\phi])^* + \partial_i h_4[\phi] = 0,$$

$$\partial_i\,w_k[\phi] = \partial_k\,w_i[\phi],$$

(38)

where, for instance, we denoted by $h_4[\phi]$ the functional dependence on ϕ. Such conditions are always satisfied for cosmological solutions with $\phi = \phi(t)$ or if $\phi = const$ (in the last case $w_i(x^k, t)$ can be any functions as follows from (34) with zero β and α_i, see (36)).

3.2 Important Special Cases

We can construct such solutions for certain special parametrizations of coefficients for ansatz (37) subjected to the condition to be a solution of (32)–(35).

3.2.1 Type 2: Solutions with $h_4^* = 0$

The equation (27) can be solved for such a case, $h_4^* = 0$, only if $\Upsilon_2 = 0$. We can consider any functions $w_i(x^k, t)$ as solutions of (28), and its equivalent (34), because the coefficients β and α_i, see (36), are zero. To find nontrivial values of n_i we can integrate (35) for $h_4^* = 0$ and any given h_3 which results in $n_i = {}^1n_k(x^i) + {}^2n_k(x^i)\int h_3 dt$. We can consider any $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ determined by (32) for a given $\Upsilon_4(x^k)$.

Summarizing the results, we conclude that this class of solutions is defined by an ansatz

$$^\eta\mathbf{g} = e^{\psi(x^k)}dx^i \otimes dx^i + h_3(x^k, t)\mathbf{e}^3 \otimes \mathbf{e}^3 + {}^0h_4(x^k)\mathbf{e}^4 \otimes \mathbf{e}^4,$$

$$\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + \left[{}^1n_k(x^i) + {}^2n_k(x^i)\int h_3 dt\right]dx^i,$$

(39)

for arbitrary generating functions $h_3(x^k, t)$, $w_i(x^k, t)$, $^0h_4(x^k)$ and integration functions $^1n_k(x^i)$ and $^2n_k(x^i)$.

The conditions (30) selecting from (39) a subclass of solutions for the Levi-Civita connection transform into the equations

$$^2n_k\left(x^i\right) = 0 \quad \text{and} \quad \partial_i{}^1n_k = \partial_k{}^1n_i,$$

$$w_i^* + \partial_i{}^0h_4 = 0 \quad \text{and} \quad \partial_i w_k = \partial_k w_i,$$

for any such $w_i(x^k, t)$ and $^0h_4(x^k)$. This class of constraints do not involve the generating function $h_3(x^k, t)$.

3.2.2 Type 3: Solutions with $h_3^* = 0$ and $h_4^* \neq 0$

Such metrics are defined by ansatz of type

$$^\eta\mathbf{g} = e^{\psi(x^k)}dx^i \otimes dx^i - {}^0h_3(x^k)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, t)\mathbf{e}^4 \otimes \mathbf{e}^4,$$

$$\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_i(x^k, t)dx^i,$$

(40)

where $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ being a solution of (32) for any given $\Upsilon_4(x^k)$. The function $h_4(x^k, t)$ must satisfy (33) which for $h_3^* = 0$ is just

$$h_4^{**} - \frac{(h_4^*)^2}{2h_4} - 2\,{}^0h_3 h_4 \Upsilon_2(x^k, t) = 0.$$

The N-connection coefficients are

$$w_i = -\partial_i \widetilde{\phi}/\widetilde{\phi}^*, \qquad n_i = {}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right) \int [1/(\sqrt{|h_4|})^3] dt,$$

when $\widetilde{\phi} = \ln |h_4^*/\sqrt{|\,{}^0h_3 h_4|}|$.

The Levi-Civita configurations for solutions (40) are selected by the conditions (30) which, for this case, are satisfied if

$${}^2n_k\left(x^i\right) = 0 \quad \text{and} \quad \partial_i\,{}^1n_k = \partial_k\,{}^1n_i,$$

and

$$\left(w_i[\widetilde{\phi}]\right)^* + w_i[\widetilde{\phi}]\left(h_4[\widetilde{\phi}]\right)^* + \partial_i h_4[\widetilde{\phi}] = 0,$$

$$\partial_i\, w_k[\widetilde{\phi}] = \partial_k\, w_i[\widetilde{\phi}].$$

Such conditions are similar to (38) but for a different relation of v-coefficients of metric to another type of generating function $\widetilde{\phi}$. They are always satisfied for cosmological solutions with $\widetilde{\phi} = \widetilde{\phi}(t)$ or if $\widetilde{\phi} = const$ (in the last case $w_i(x^k, t)$ can be any functions as follows from (34) with zero β and α_i, see (36)).

3.2.3 Type 4: Solutions with $\phi = const$

If we fix $\phi = \phi_0 = const$ in (36), but $h_3^* \neq 0$ and $h_4^* \neq 0$, we can express the general solutions of (32)–(35) in the form

$$
\begin{aligned}
{}^n\mathbf{g} = &\, e^{\psi(x^k)} dx^i \otimes dx^i - {}^0h^2 \left[f^*\left(x^i, t\right)\right]^2 |\varsigma_\Upsilon\left(x^i, t\right)| \mathbf{e}^3 \otimes \mathbf{e}^3 \\
&+ f^2\left(x^i, t\right) \mathbf{e}^4 \otimes \mathbf{e}^4, \\
\mathbf{e}^3 = &\, dt + w_i(x^k, t) dx^i, \qquad \mathbf{e}^4 = dy^4 + n_k\left(x^i, t\right) dx^i,
\end{aligned}
\tag{41}
$$

where ${}^0h = const$, $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ being a solution of (32) for any given $\Upsilon_4(x^k)$, and $\varsigma_\Upsilon(x^i, t) = \varsigma_{4[0]}(x^i) - \frac{h_0^2}{16}\int \Upsilon_2(x^k, t)[f^2(x^i, t)]^2 dt$. The N-connection coefficients $N_i^3 = w_i(x^k, t)$ and $N_i^4 = n_i(x^k, t)$ are

$$w_i = -\frac{\partial_i \varsigma_\Upsilon(x^k, t)}{\varsigma_\Upsilon^*(x^k, t)} \tag{42}$$

and

$$n_k = {}^1n_k(x^i) + {}^2n_k(x^i) \int \frac{[f^*(x^i, t)]^2}{[f(x^i, t)]^2} \varsigma_\Upsilon\left(x^i, t\right) dt. \tag{43}$$

We must take $\varsigma_{4[0]}(x^i) = \pm 1$ if $\varsigma_\Upsilon(x^i, t) = \pm 1$ for $\Upsilon_2 \to 0$. In such a case, the functions $h_3 = -\,^0 h^2\,[f^*(x^i, t)]^2$ and $h_4 = f^2(x^i, t)$ satisfy (33) written in the form $\sqrt{|h_3|} = \,^0 h(\sqrt{|h_4|})^*$, which is compatible with the condition $\phi = \phi_0$.

The subclass of solutions for the Levi-Civita connection with ansatz of type (41) is subjected additionally to the conditions (30). For instance, we can chose that $^2 n_k(x^i) = 0$ and $^1 n_k(x^i)$ are any functions satisfying the conditions $\partial_i\,^1 n_k = \partial_k\,^1 n_i$. The constraints on values $w_i = -\partial_i \varsigma_\Upsilon / \varsigma_\Upsilon^*$ result in constraints on ς_Υ, which is determined by Υ_2 and f,

$$
\begin{aligned}
&(w_i[\varsigma_\Upsilon])^* + w_i[\varsigma_\Upsilon]\,(h_4[\varsigma_\Upsilon])^* + \partial_i h_4[\varsigma_\Upsilon] = 0, \\
&\partial_i\, w_k[\varsigma_\Upsilon] = \partial_k\, w_i[\varsigma_\Upsilon],
\end{aligned}
\tag{44}
$$

where, for instance, we denoted by $h_4[\varsigma_\Upsilon]$ the functional dependence on ς_Υ. Such conditions are always satisfied for cosmological solutions with $f = f(t)$. For $\widehat{\mathbf{D}}$, if $\Upsilon_2 = 0$ and $\phi = const$, the coefficients $w_i(x^k, t)$ can be arbitrary functions (we can fix $\varsigma_\Upsilon = 1$, which does not impose a functional dependence of w_i on ς_Υ) as follows from (34) with zero β and α_i, see (36). To generate solutions for ∇ such w_i must be additionally constrained following formulas (44) re-written for $w_i[\varsigma_\Upsilon] \to w_i(x^k, t)$ and $h_4[\varsigma_\Upsilon] \to h_4(x^i, t)$.

We note that any solution $\mathbf{g} = \{g_{\alpha'\beta'}(u^{\alpha'})\}$ of the Einstein equations (23) and/or (1) with Killing symmetry $\partial/\partial y$ (for local coordinates in the form $y^3 = t$ and $y^4 = y$) can be parametrized in a form derived in this section. Using frame transforms of type $e_\alpha = e_\alpha^{\alpha'} e_{\alpha'}$, with $\mathbf{g}_{\alpha\beta} = e_\alpha^{\alpha'} e_\beta^{\beta'} g_{\alpha'\beta'}$, for any $\mathbf{g}_{\alpha\beta}$ (2), we relate the class of such (inhomogeneous) cosmological solutions, for instance, to the family of metrics of type (31).[14]

Following recent developments from [17], we can construct 'non-Killing' solutions depending on all coordinates. Such general classes of solutions can be parametrized in the form

$$
\begin{aligned}
\mathbf{g} &= +g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j, t, y)h_a(x^k, t)\mathbf{e}^a \otimes \mathbf{e}^a, \\
\mathbf{e}^3 &= dy^3 + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_i(x^k, t)dx^i,
\end{aligned}
\tag{45}
$$

for any ω for which $\mathbf{e}_k \omega = \partial_k \omega + w_k \omega^* + n_k \partial \omega / \partial y = 0$, when (45) with $\omega^2 = 1$ is of type (2).

Finally, we note that the metrics constructed above define general cosmological solutions of Einstein equations for any type of sources $\varkappa T_{\beta\delta}$ which can be parametrized[15] in a formally diagonalized form Υ_γ (25), with respect to a nonholonomic frame of reference.

4 Examples of Cosmological Models with Off-Diagonal Metrics

The goal of this section is to analyze explicit examples of generic off-diagonal cosmological solutions with $\Upsilon_4 = 0$ but, in general, with non-vanishing $\Upsilon_2(t)$. Such solutions are constructed as examples of metrics (37), (39), (40) and (41). The new classes of cosmological metrics have respective 'diagonal' limits to conformal and/or frame transforms of metrics (17), (18), (20), (22).

---

[14]We have to solve certain systems of quadratic algebraic equations and define some $e_\alpha^{\alpha'}(u^\beta)$, choosing a convenient system of coordinates $u^{\alpha'} = u^{\alpha'}(u^\beta)$.

[15]Using chains of nonholonomic frame transforms, this is possible for 'almost' all physically important energy-momentum tensors.

We consider a particular parametrization of metrics of type (2) when

$$
\begin{aligned}
^\eta\mathbf{g} &= \eta_i(x^k, t) \, {}^\circ g_i(x^k, t)dx^i \otimes dx^i + \eta_a(t) \, {}^\circ h_a(t)\mathbf{e}^a \otimes \mathbf{e}^a, \\
\mathbf{e}^3 &= dt + \eta_i^3(t) \, {}^\circ w_i(t)dx^i, \qquad \mathbf{e}^4 = dy^4 + \eta_i^4(t) \, {}^\circ n_i(t)dx^i,
\end{aligned}
\tag{46}
$$

or (for an alternative parametrization)

$$
\mathbf{e}^3 = dt + \left[\eta_i^3(t) + {}^\circ w_i(t)\right] dx^i, \qquad \mathbf{e}^4 = dy^4 + \left[\eta_i^4(t) + {}^\circ n_i(t)\right] dx^i,
$$

when (for some constructions) $g_1 = \eta_1(x^k, t) \, {}^\circ g_1(x^k, t) = 1$ and $g_2 = \eta_1(x^k, t) \, {}^\circ g_2(x^k, t) = 1$ are trivial solutions of (26), and (32), with $\Upsilon_4 = 0$. For homogeneous configurations, we can always introduce such coordinates when the coefficients of metrics do not depend on space variables (a formal presence of radial and angular variables may exist, for instance, in spherical coordinates, like in the case of FRW metric (13)) The polarization functions $\eta_a(t)$ and $\eta_i^a(t)$ can be chosen in a form which allows us to generate homogeneous configurations (i.e. solutions not depending on space coordinates) with $h_a(t) = \eta_a(t) \, {}^\circ h_a(t)$ and $w_i(t) = \eta_i^3(t) \, {}^\circ w_i(t)$, $n_i(t) = \eta_i^4(t) \, {}^\circ n_i(t)$.

4.1 Nonholonomic FRW, Bianchi, Kasner and Gödel Type Configurations

4.1.1 N-anholonomic FRW Generalizations

Solutions of Type 1 We chose ${}^\circ g_i = 1$, ${}^\circ h_3(t) = -a^{-2}(t)$, ${}^\circ h_4 = 1$, where $a(t)$ is determined by (15) and (16) for the FRW model. A class of anisotropic and inhomogeneous solutions parametrized by metrics of type (37) is generated by data

$$
\begin{aligned}
g_1 = g_2 &= 1, \qquad h_3 = \pm \frac{|\phi^*(x^i, t)|}{\Upsilon_2(x^i, t)}, \\
h_4 &= {}^0 h_4(x^k) \pm 2 \int \frac{(\exp[2\phi(x^k, t)])^*}{\Upsilon_2(x^i, t)} dt, \\
w_i &= -\partial_i \phi / \phi^*, \qquad n_i = {}^1 n_k(x^i) + {}^2 n_k(x^i) \int [h_3 / (\sqrt{|h_4|})^3] dt.
\end{aligned}
$$

Off-diagonal cosmological metrics depending only on variable t can be extracted by gravitational polarizations

$$
\begin{aligned}
\eta_i &= 1, \qquad \eta_3 = a^2(t)|\phi^*(t)| / \Upsilon_2(t), \qquad \eta_4 = 1 \pm 2 \int \frac{(\exp[2\phi(t)])^*}{\Upsilon_2(t)} dt, \\
\eta_i^3(t) &= 0, \qquad \eta_i^4(t) = {}^1 n_k + {}^2 n_k \int [\eta_3(t) / (\sqrt{|\eta_4(t)|})^3] dt,
\end{aligned}
\tag{47}
$$

with some ${}^1 n_k = const$, ${}^2 n_k = const$, when $w_i(t) = \eta_i^3(t) + {}^\circ w_i(t) = 0$, for ${}^\circ w_i(t) = 0$, and $n_i(t) = \eta_i^4(t) + {}^\circ n_i(t)$, for ${}^\circ n_i(t) = 0$. The factor $a^2(t)$ can be included into a generating function $\phi(t)$, or into a source $\Upsilon_2(t)$ (we can say that it is nonholonomically modelled by such a generating function, alternatively, effective source). The off-diagonal metric

$$
\begin{aligned}
^\eta\mathbf{g} &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \frac{|\phi^*(t)|}{\Upsilon_2(t)} dt \otimes dt \\
&\quad + \left[1 \pm 2 \int \frac{(\exp[2\phi(t)])^*}{\Upsilon_2(t)} dt\right] \left[dy^4 + \eta_i^4(t)dx^i\right] \otimes \left[dy^4 + \eta_i^4(t)dx^i\right],
\end{aligned}
\tag{48}
$$

with $\eta_i^4(t)$ computed following formulas (47) defines a class of cosmological solutions of the Einstein equations (23) for $\widehat{\mathbf{D}}$ with a source $\Upsilon^{\alpha}{}_{\beta} = \mathrm{diag}[\Upsilon_{\gamma}; \Upsilon_1 = \Upsilon_2 = \Upsilon_2(t); \Upsilon_3 = \Upsilon_4 = 0]$. We can extract Levi-Civita configurations if we fix, for instance, the integration constants $^1 n_k = {}^2 n_k = 0$.

For gravitational polarizations with certain smooth limits

$$\frac{|\phi^*(t)|}{\Upsilon_2(t)} \to -a^{-2}(t), \qquad \int \frac{(\exp[2\,\phi(t)])^*}{\Upsilon_2(t)} dt \to 0, \qquad \eta_i^4(t) \to 0$$

(this can be satisfied by a corresponding choosing of functions $\phi(t)$, $\Upsilon_2(t)$ and integration constants $^1 n_k$, $^2 n_k$), the solutions (48) transform not just into the FRW metric (13) but into its conformal transform (with multiplication on factor $a^{-2}(t)$). Such off-diagonal cosmological solutions have certain similarities (for small nonholonomic deformations) to conformally transformed FRW solutions if $\phi(t)$, $\Upsilon_2(t)$ are fixed to mimic $a(t)$ as in (15) and (16) for the Hubble parameter.

Solutions of Type 2 As for the type 1, we also take $^{\circ}g_i = 1$, $^{\circ}h_3(t) = -a^{-2}(t)$, $^{\circ}h_4 = 1$ but generate anisotropic and inhomogeneous solutions parametrized by metrics of type (39)

$$
\begin{aligned}
{}^{\eta}\mathbf{g} &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + h_3(x^k, t)\mathbf{e}^3 \otimes \mathbf{e}^3 + {}^0 h_4(x^k)\mathbf{e}^4 \otimes \mathbf{e}^4, \\
\mathbf{e}^3 &= dt + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + \left[{}^1 n_k\left(x^i\right) + {}^2 n_k\left(x^i\right) \int h_3 dt \right] dx^i,
\end{aligned}
\tag{49}
$$

for arbitrary generating functions $h_3(x^k, t)$, $w_i(x^k, t)$, $^0 h_4(x^k)$ and integration functions $^1 n_k(x^i)$ and $^2 n_k(x^i)$, when $\Upsilon_2 = 0$. The polarization functions are

$$\eta_i = 1, \qquad \eta_3 = a^2(t) h_3(x^k, t), \qquad \eta_4 = {}^0 h_4(x^k),$$

$$\eta_i^3 = w_i(x^k, t), \qquad \eta_i^4 = {}^1 n_k\left(x^i\right) + {}^2 n_k\left(x^i\right) \int h_3 dt,$$

with $^{\circ}w_i(t) = 0$ and $^{\circ}n_i(t) = 0$. Such cosmological solutions are anisotropic and inhomogeneous and constructed as nonholonomic deformations of a conformally transformed (with multiplication on factor $a^{-2}(t)$) FRW metric (13). We can prescribe polarization functions $\eta_3(x^k, t)$ when $h_3 = \eta_3\, {}^{\circ}h_3(t) \to -a^{-2}(t)$ for $\eta_3 \to 1$.

The conditions (30) selecting Levi-Civita configurations transform into equations

$$
\begin{aligned}
&{}^2 n_k\left(x^i\right) = 0 \quad \text{and} \quad \partial_i\, {}^1 n_k = \partial_k\, {}^1 n_i, \\
&w_i^* + \partial_i\, {}^0 h_4 = 0 \quad \text{and} \quad \partial_i\, w_k = \partial_k\, w_i.
\end{aligned}
\tag{50}
$$

Such constraints can be satisfied for any generating function $h_3(x^k, t)$ but impose additional constraints on N-coefficients $w_i(x^k, t)$.

Off-diagonal metrics of type (49) can be generated to be only with time like dependence of coefficients, when $h_3 = h_3(t)$, $w_i = w_i(t)$ and $n_i(t)$ are determined with some constant values of $^0 h_4$, $^1 n_k$, $^2 n_k$. Such conditions are for the Levi-Civita configurations if $w_i = const$. This defines vacuum solutions of the Einstein equations. They transform conformally and nonholonomically a FRW universe into certain vacuum Einstein configurations which (in this particular case) can be diagonalized by coordinate transforms.

Solutions of Type 3   It is not possible to construct off-diagonal generalizations of type (40), with coordinate $y^3 = t$, for FRW universes, when $h_3 = -\,^0h_3(x^k)$ does not depend on t and $^\circ g_i = 1$, $^\circ h_3(t) = -a^{-2}(t)$, $^\circ h_4 = 1$. For such solutions, we can not obtain $h_3 \sim a^{-2}(t)$ in the limit of trivial polarizations. Considering inhomogeneous metrics of type (45) with $\omega = \omega(x^i, t, y)$, where $a^{-2}(t)$ can be included into ω^2, we model general inhomogeneous solutions obtained via "vertical" conformal transforms and further nonholonomic deformations. The metrics are of type

$$
\begin{aligned}
^\eta\mathbf{g} &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \\
&\quad + a^{-2}(t)\widetilde{\omega}^2(x^i, t, y)\left[-\,^0h_3(x^k)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, t)\mathbf{e}^4 \otimes \mathbf{e}^4\right],
\end{aligned}
\tag{51}
$$

$$
\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_i(x^k, t)dx^i,
$$

where $\omega = \widetilde{\omega}/a$ is a solution of $\partial_k\omega + w_k\omega^* + n_k\partial\omega/\partial y = 0$. The equation (33) for $h_3^* = 0$ transforms into

$$
h_4^{**} - \frac{(h_4^*)^2}{2h_4} - 2\,^0h_3h_4\Upsilon_2(x^k, t) = 0.
\tag{52}
$$

Integrating two times on t, we can define $h_4(x^k, t)$ for any given $^0h_3(x^k)$ and $\Upsilon_2(x^k, t)$. Then we can compute the N-connection coefficients following formulas

$$
w_i = -\partial_i\widetilde{\phi}/\widetilde{\phi}^*, \qquad n_i = \,^1n_k\left(x^i\right) + \,^2n_k\left(x^i\right)\int [1/(\sqrt{|h_4|})^3]dt,
\tag{53}
$$

for $\widetilde{\phi} = \ln|h_4^*/\sqrt{|\,^0h_3h_4|}|$.

The Levi-Civita configurations are extracted from the set of solutions (51) if

$$
^2n_k\left(x^i\right) = 0 \quad \text{and} \quad \partial_i\,^1n_k = \partial_k\,^1n_i,
$$

and

$$
\begin{aligned}
&\left(w_i[\widetilde{\phi}]\right)^* + w_i[\widetilde{\phi}]\left(h_4[\widetilde{\phi}]\right)^* + \partial_ih_4[\widetilde{\phi}] = 0, \\
&\partial_i\,w_k[\widetilde{\phi}] = \partial_k\,w_i[\widetilde{\phi}].
\end{aligned}
$$

Such conditions are always satisfied for cosmological solutions with $\widetilde{\phi} = \widetilde{\phi}(t)$ or if $\widetilde{\phi} = const$ (in the last case $w_i(x^k, t)$ can be any functions as follows from (34) with zero β and α_i, see (36); in such cases, we must take $\Upsilon_2 = 0$).

The first subclass of metrics (51) (with solutions depending only on t, with nontrivial $\omega(t, y)$, for constant $^0h_3$ and ^1n_k; $^2n_k = 0$ and nontrivial $\Upsilon_2(t)$) are parametrized in the form

$$
\begin{aligned}
^\eta\mathbf{g} &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + a^{-2}(t)\widetilde{\omega}^2(t, y) \\
&\quad \times \left[-\,^0h_3dt \otimes dt + h_4(t)(dy^4 + \,^1n_idx^i) \otimes (dy^4 + \,^1n_idx^i)\right],
\end{aligned}
\tag{54}
$$

where $h_4(t)$ is any solution of $h_4^{**} - \frac{(h_4^*)^2}{2h_4} - 2\,^0h_3h_4\Upsilon_2(t) = 0$.

The second subclass of metrics (51) contains arbitrary functions $w_i(t)$ but for $\Upsilon_2(t) = 0$ and $h_4(t)$ is any solution of $h_4^{**} - (h_4^*)^2/(2h_4) = 0$. For constant $^0h_3$ and ^1n_k and $^2n_k = 0$, we get

$$
^\eta\mathbf{g} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + a^{-2}(t)\widetilde{\omega}^2(t, y)
$$

$$\times [- {}^0h_3(dt + w_i(t)dx^i) \otimes (dt + w_i(t)dx^i)$$
$$+ h_4(t)(dy^4 + {}^1n_i dx^i) \otimes (dy^4 + {}^1n_i dx^i)]. \tag{55}$$

Both types of solutions (54) and (55) are for the Levi-Civita connection. They can be generalized for the canonical d-connection by introducing nontrivial constants ${}^2n_k$ in the formulas for N-connection coefficients, $N_i^4 = n_i(t)$, see (53), with $h_4(t)$ respectively computed as in (52), when $\Upsilon_2 = \Upsilon_2(t)$ and or $\Upsilon_2 = 0$.

For the metric (51), the polarization functions are ${}^0h_3(x^k)$

$$\eta_i = 1, \qquad \eta_3 = \widetilde{\omega}^2(x^i, t, y) \, {}^0h_3(x^k), \qquad \eta_4 = a^{-2}(t)\widetilde{\omega}^2(x^i, t, y) \, h_4(x^k, t),$$

$$\eta_i^3 = w_i(x^k, t), \qquad \eta_i^4 = {}^1n_k(x^i) + {}^2n_k(x^i) \int h_3 dt,$$

with ${}^\circ w_i(t) = 0$ and ${}^\circ n_i(t) = 0$. We have to eliminate respectively the dependencies on x^i in these formulas in order to compute the gravitational polarizations for the metrics (54) and (55). For simplicity, we omit the formulas for this class of solutions. Such classes of metrics limit to the conformally transformed FRW one (with conformal factor $a^{-2}(t)$) if $\eta_\alpha \to 1$ and $\eta_i^a \to 0$.

Solutions of Type 4 If we fix $\phi = \phi_0 = const$ in (36), but $h_3^* \neq 0$ and $h_4^* \neq 0$, we have to reparametrize the sets of generating and integration functions in a different form. The solutions (41) with ${}^\circ g_i = 1$, ${}^\circ h_3(t) = -a^{-2}(t)$, ${}^\circ h_4 = 1$ are of type

$${}^\eta\mathbf{g} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$
$$- {}^0h^2 \left[f^*\left(x^i, t\right) \right]^2 |\varsigma_\Upsilon\left(x^i, t\right)| \mathbf{e}^3 \otimes \mathbf{e}^3 + f^2\left(x^i, t\right) \mathbf{e}^4 \otimes \mathbf{e}^4, \tag{56}$$
$$\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_k\left(x^i, t\right)dx^i,$$

where

$$\varsigma_\Upsilon = 1 - \frac{h_0^2}{16} \int \Upsilon_2(x^k, t)\left[f^2\left(x^i, t\right) \right]^2 dt,$$

$$w_i = -\partial_i \varsigma_\Upsilon\left(x^k, t\right) / \varsigma_\Upsilon^*\left(x^k, t\right),$$

$$n_k = {}^1n_k\left(x^i\right) + {}^2n_k\left(x^i\right) \int \frac{[f^*(x^i, t)]^2}{[f(x^i, t)]^2} \varsigma_\Upsilon\left(x^i, t\right) dt.$$

The Levi-Civita configurations are chosen when ${}^2n_k = 0$ and ${}^1n_k(x^i)$ are any functions with $\partial_i \, {}^1n_k = \partial_k \, {}^1n_i$. The constraints for $w_i = -\partial_i \varsigma_\Upsilon / \varsigma_\Upsilon^*$ are similar to (44).

We obtain from (56) generic off-diagonal homogeneous cosmological solutions if we consider metrics with dependence only on t. They can be written in the form

$${}^\eta\mathbf{g} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - {}^0h^2 \left[f^*(t) \right]^2 |\varsigma_\Upsilon(t)| \mathbf{e}^3 \otimes \mathbf{e}^3 + f^2(t) \mathbf{e}^4 \otimes \mathbf{e}^4,$$
$$\mathbf{e}^3 = dt + w_i(t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_k(t) dx^i, \tag{57}$$

where, for constant ${}^1n_k$, ${}^2n_k$ and ${}^0h$, and $\varsigma_\Upsilon(t) = 1 - \frac{h_0^2}{16} \int \Upsilon_2(t)[f^2(t)]^2 dt$,

$$w_i = -\partial_i \varsigma_\Upsilon(t) / \varsigma_\Upsilon^*(t), \qquad n_k = {}^1n_k + {}^2n_k \int \frac{[f^*(x^i, t)]^2}{[f(x^i, t)]^2} \varsigma_\Upsilon(t) dt.$$

For simplicity, we omit the explicit formulas for gravitational polarizations (they can be computed similarly to those given for above examples).

Taking $^2 n_k = 0$ for (57) and imposing constraints of type (44) for $w_i(t)$, we generate new classes of cosmological solutions in general relativity. For trivial gravitational polarizations and vanishing N-connection coefficients, such metrics result in the conformal transform (with factor $a^{-2}(t)$) of the FRW metric (13).

4.1.2 Locally Anisotropic Bianchi Spacetimes

We speculate how a Bianchi metric $_B\mathbf{g}_{\alpha''\beta''}$ (18) can be generalized into locally anisotropic solutions. Taking any set of coefficients $_B g_{\underline{\alpha}\underline{\beta}}(t)$, we have to construct certain frame transform, $\overset{\circ}{_B}\mathbf{g}_{\alpha\beta} = {_B}e^{\underline{\alpha}}_{\alpha''}\,{_B}e^{\underline{\beta}}_{\beta''}\,{_B}g_{\underline{\alpha}\underline{\beta}}$,[16] when $\overset{\circ}{_B}\mathbf{g}_{\alpha\beta}$ is parametrized as a prime metric

$$\overset{\circ}{_B}\mathbf{g} = {}^\circ g_i dx^i \otimes dx^i + {}^\circ h_a(t)\mathbf{e}^a \otimes \mathbf{e}^a, \qquad \mathbf{e}^3 = dt + {}^\circ w_i(t)dx^i, \qquad \mathbf{e}^4 = dy^4 + {}^\circ n_i(t)dx^i.$$

Introducing corresponding gravitational η-polarizations, we construct nonholonomic deformations $\overset{\circ}{_B}\mathbf{g} \to \overset{\eta}{_B}\mathbf{g}$, where the target metric can be any one of type (45), or (for $\omega^2 = 1$) of type (2). In general, such configurations are locally anisotropic and inhomogeneous when the solutions depend on all coordinates.

We can derive generic off-diagonal solutions with the coefficients depending only on t, or being some constants, as we proved in details in previous sections. The main difference is that for trivial gravitational polarizations such metrics describe frame/conformal transforms of Bianchi spacetimes and not of conformal transforms of FRW metrics. Explicit constructions depend on "structure constants" (19) for the prime metric and are, in general, different for four types of off-diagonal generalizations. The length of this paper does not allow us to provide such details and analyze possible applications in cosmology.

4.1.3 Off-Diagonal Kasner Type Metrics

The data for the primary metric are taken in the form $^\circ g_1 = 1$, $^\circ g_2 = t^{2(p_3 - p_1)}$, $^\circ h_3 = -t^{-2p_1}$, $^\circ h_4 = t^{2(p_2 - p_1)}$ and $^\circ N_i^a = 0$ with constants p_1, p_2 and p_3 as in the metric (20). For simplicity, we analyze in this section solutions with $p_3 = p_1$ and consider an example when a Kasner universe is generalized to locally anisotropic configurations of type 4 characterized by gravitational polarizations

$$\eta_i = 1, \qquad \eta_3 = {}^0 h^2 \left[f^* \left(x^i, t \right) \right]^2, \qquad \eta_4 = f \left(x^i, t \right),$$
$$\eta_i^3 = w_i(x^i, t), \qquad \eta_i^4 = n_i(x^k, t).$$

For $h_a = \eta_a\,{}^\circ h_a$ and $N_i^a = \eta_i^a + {}^\circ N_i^a$, the target metric is of type (41) generated for $\Upsilon_2 = 0$,

$$\overset{\eta}{}\mathbf{g} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$
$$- {}^0 h^2 \left[f^* \left(x^i, t \right) \right]^2 t^{-2p_1} \mathbf{e}^3 \otimes \mathbf{e}^3 + f^2 \left(x^i, t \right) t^{-2p_1} \mathbf{e}^4 \otimes \mathbf{e}^4, \qquad (58)$$
$$\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \qquad \mathbf{e}^4 = dy^4 + n_k \left(x^i, t \right) dx^i,$$

---

[16]We have to solve certain quadratic algebraic equations in order to define frame coefficients depending on coordinate t, or being some constants.

where we can consider arbitrary $w_i = w_i(x^i, t)$ and

$$n_k = {}^1 n_k(x^i) + {}^2 n_k(x^i) \int \frac{[f^*(x^i, t)]^2}{[f(x^i, t)]^2} dt.$$

The coefficients h_a are solutions of the Einstein equations for the canonical d-connection, see (27) (and/or, equivalently, (33)) written in the form $\sqrt{|h_3|} = {}^0 h(\sqrt{|h_4|})^*$ for arbitrary generating function $f(x^i, t)$ if $p_2 = p_1$. We have to impose additional constraints on $f(x^i, t)$ if the last condition is not satisfied.

In the limit of trivial polarizations, the metric (58) results into a conformal transform (with factor t^{2p_1}) of Kasner solution (20). In general, such a prime metric is not a solution of the Einstein equations for the Levi-Civita connection but it is possible to chose gravitational polarizations generating vacuum off-diagonal Einstein fields even the conditions of type (21) are not satisfied. Such target metrics may be stable (it is necessary an additional analysis of stability properties for any explicit case).

We can eliminate dependence on space coordinates and generate solutions of type

$$\begin{aligned}
{}^\eta \mathbf{g} &= dx^1 \otimes dx^1 + dx^2 \otimes dx^2 \\
&\quad - {}^0 h^2 \left[f^*(t) \right]^2 t^{-2p_1} \mathbf{e}^3 \otimes \mathbf{e}^3 + f^2(t) \, t^{-2p_1} \mathbf{e}^4 \otimes \mathbf{e}^4,
\end{aligned}$$

$$\mathbf{e}^3 = dt + w_i(t) dx^i, \qquad \mathbf{e}^4 = dy^4 + n_k(t) dx^k,$$

for arbitrary $w_i = w_i(t)$ and constant ${}^1 n_k$ and ${}^2 n_k$, when $n_k = {}^1 n_k + {}^2 n_k \int dt [f^*(t)]^2 / [f(t)]^2$. To extract Levi-Civita configurations we must fix ${}^2 n_k = 0$ and impose constraints of type (44) on $w_i(t)$.

In a similar form, we can construct nonholonomic deformations of the Kasner universes of types 1–3 and/or to generalize them to solutions of type (45). The corresponding target metrics may be with "gravitational chaos" or constrained nonholonomically to became stable. We omit such details in this work.

4.1.4 Rotating Cosmologies with Local Anisotropy

We can chose the prime's metric data to be given by the Gödel solution (22) when $\,^\circ g_i = {}_G g_i(x)$, $\,^\circ h_a = {}_G h_a$ and $\,^\circ N_i^a = {}_G N_i^a$, with local coordinates $x^i = (x, z)$ and $y^a = (t, y)$. Considering polarizations

$$\eta_1 = e^{\psi x, z)}, \qquad \eta_2 = 2 e^{\psi(x,z) - 2x}, \qquad \eta_3 = \eta_3(x, z, t), \qquad \eta_4 = \eta_4(x, z, t),$$

$$\eta_i^3 = w_i(x, z, t), \qquad \eta_i^4 = n_k(x, z, t),$$

for $g_i = \eta_i \,^\circ g_i$, $\,^\circ h_a = \eta_a \,^\circ h_a$, $N_i^a = \eta_i^3 + {}^\circ N_i^a$, when $N_i^3 = w_i(x, z, t)$ and $N_i^4 = n_i(x, z, t)$, we generate metrics of type

$$\begin{aligned}
{}^\eta_G \mathbf{g} &= {}_G a^2 [e^{\psi x, z)} (dx \otimes dx + dz \otimes dz) \\
&\quad - \eta_3(x, z, t)(dt + w_i(x, z, t)dx^i) \otimes (dt + w_i(x, z, t)dx^i) \\
&\quad + \eta_4(x, z, t)(dy + n_i(x, z, t)dx^i) \otimes (dy + n_i(x, z, t)dx^i)].
\end{aligned} \tag{59}$$

Choosing a source determined by cosmological constant $\Upsilon_2 = \Upsilon_4 = {}_G \lambda$, we construct a class of type 1 exact solutions if

$$\ddot{\psi} + \psi'' = 2 \,_G \lambda, \qquad \eta_4^* = 2 \,_G a^2 \, \eta_3 \eta_4 \,_G \lambda / \phi^*,$$

$$\beta w_i + \alpha_i = 0, \qquad n_i^{**} + \gamma n_i^* = 0,$$

with the coefficients

$$\alpha_i = {}_G a^2\, \eta_4^*\, \partial_i \phi, \quad \beta = {}_G a^2\, \eta_4^*\, \phi^*, \quad \text{for } \phi(x,z,t) = \ln \left| \frac{\eta_4^*}{\sqrt{|{}_G a^2\, \eta_3 \eta_4|}} \right|,$$

$$\gamma = \left(\ln |{}_G a^2\, \eta_4|^{3/2} / |{}_G a^2\, \eta_3| \right)^*.$$

Such solutions are derived directly from prime Gödel metrics and have limits to rotating universes for trivial polarizations.

In explicit form, we can model locally anisotropic and inhomogeneous models with rotation when $h_4^* \neq 0$; for ${}_G \lambda \neq 0$, we get $\phi^* \neq 0$. We obtain (computing consequently for a prescribed generating function $\phi(x,z,t)$)

$$\eta_1 = e^{\psi x,z}, \qquad \eta_2 = 2 e^{\psi(x,z) - 2x}, \qquad \eta_3 = \pm\, {}_G \lambda^{-1} \times |\phi^*(x,z,t)|,$$
$$\eta_4 = {}^0 \eta_4(x,z) \pm 2\, {}_G \lambda^{-1} \times \exp[2\, \phi(x,z,t)] dt,$$
$$w_i = -\partial_i \phi / \phi^*, \tag{60}$$
$$n_i = {}^1 n_k(x,z) + {}^2 n_k(x,z) \int [\eta_3(x,z,t)/(\sqrt{|\eta_4(x,z,t)|})^3] dt.$$

The metric (59) with coefficients (60) is an explicit example of solutions of type (37) when the source is determined by a cosmological constant. It defines a rotating cosmology additionally imbedded into nontrivial gravitational backgrounds, which (in general) are locally anisotropic and inhomogeneous. We can impose restrictions of type (30) and select Levi-Civita configurations. It is possible also to construct models with anisotropic rotation when such solutions do not depend on x^i, but only on t, or with generalizations of the Gödel model determined by off-diagonal solutions of type 2–4.

4.2 Modeling Anisotropic De Sitter Configurations

We can generate cosmological solutions when the coefficients $g_i(x^k, t)$ in (2) depend explicitly on variable t. Let us consider a conformal factor $q(x^k, t)$ when

$$^q\mathbf{g} = q^2(x^k, t)[\eta_i(x^k)\, {}^\circ g_i dx^i \otimes dx^i$$
$$+ \eta_3(x^k, t)\, {}^\circ h_3(t) \mathbf{e}^3 \otimes \mathbf{e}^3 + {}^\circ h_4(x^k) \mathbf{e}^4 \otimes \mathbf{e}^4], \tag{61}$$
$$\mathbf{e}^3 = dt + w_i(x^k, t) dx^i, \qquad \mathbf{e}^4 = dy^4 + n_i(x^k, t) dx^i.$$

By straightforward computations, we can prove that the Riemann and Ricci tensor for an arbitrary metric compatible d-connection, see details in [21], do not change under transform $^\eta\mathbf{g} = [g_{ij}, h_{ab}, N_i^a] \to {}^q\mathbf{g} = [q^2 g_{ij}, q^2 h_{ab}, N_i^a]$ if

$$\mathbf{e}_i q = \partial q / \partial x^i - w_i q^* = 0. \tag{62}$$

For an ansatz of form (61), the Einstein equations for the canonical d-connection (23) with source (25) is equivalent to (26)–(29) and additional equations (62) for $q(x^k, t)$.

We search a subclass of inhomogeneous solutions when $q = \widetilde{q}(x^k, t) a^2(t)$, with a prime metric ${}^\circ g_i = 1$, ${}^\circ h_3 = a^{-2}(t)$, ${}^\circ h_4 = {}^\circ h_4(x^k)$, ${}^\circ N_i^a = 0$, and η-polarizations chosen such a way that data

$$g_i = \eta_i(x^k), \qquad h_3 = \eta_3(x^k, t)\, {}^\circ h_3(t), \qquad h_4 = {}^\circ h_4(x^k),$$

$$w_i = \eta_i^3(x^k, t) + \,^\circ N_i^3, \qquad n_i = \eta_i^4(x^k, t) + \,^\circ N_i^4,$$

generate a metric (solution of the Einstein equations) of type (49) with arbitrary generating functions w_i and h_3. For trivial polarizations, when $\eta_\alpha \to 1$ and $\eta_i^a \to 0$, and $\widetilde{q} = \,^\circ h_4 = 1$, the metric (61) transform into the FRW metric (17) if we take $a^2(t)$ to be determined as a solution of the Friedmann equations (15) and (16).

The goal of this section is to prove that by corresponding nonholonomic distributions (constraints) on inhomogeneous metrics we can model de Sitter like (exponential on t) cosmological solutions of vacuum Einstein equations. In such a model, a function $a(t) = a_0 e^{Ht}$ contains the Hubble constant H as an experimental parameter determining a class of nonholonomic constraints and related conformal transform with factor $q = \widetilde{q}(x^k, t) a_0 e^{Ht}$, when (62) are

$$\partial \ln |\widetilde{q}| / \partial x^i - 2w_i H = 0.$$

Parameterizing $w_i = \,^t w_i(t) + \,^s w_i(x^k)$ when $\partial_k \,^s w_i = \partial_i \,^s w_k$, we solve this equation and satisfy the Levi-Civita conditions (50) if

$$\eta_i^4 = \,^1 n_i(x^k), \quad \text{with } \partial_k \,^1 n_i = \partial_i \,^1 n_k,$$

$$^s w_k = -\partial_k \,^\circ h_4 \quad \text{and} \quad \ln |\widetilde{q}(x^k, t)| = -2H \int \,^s w_i(t) dx^i - \,^\circ h_4(x^i).$$

Putting together the above formulas in (61), we get a class of inhomogeneous off-diagonal cosmological solutions with local anisotropy,

$$^q\mathbf{g} = a_0^2 e^{2Ht} \exp\left[-2H \int \,^t w_i(t) dx^i - \,^\circ h_4(x^i)\right][dx \otimes dx + dz \otimes dz$$
$$+ \eta_3(x^k, t) a_0^{-2} e^{-2Ht} \mathbf{e}^3 \otimes \mathbf{e}^3 + \,^\circ h_4(x^k) \mathbf{e}^4 \otimes \mathbf{e}^4], \qquad (63)$$
$$\mathbf{e}^3 = dt + [\,^t w_i(t) + \,^s w_i(x^k)] dx^i, \qquad \mathbf{e}^4 = dy^4 + \,^1 n_i(x^k) dx^i.$$

This metric models a de Sitter like expansion with arbitrary generating/integration functions $(\,^t w_i(t), \,^s w_i(x^k), \eta_3(x^k, t), \,^\circ h_4(x^i) \,^1 n_i(x^k))$ and constants a_0 and H which must be chosen following certain boundary/symmetry and other physical superpositions to satisfy the experimental data. We can fix such polarization functions (i.e. generating/integration functions) when a very short accelerated "inflationary" stage is dominate by a locally anisotropic and inhomogeneous vacuum solution of the Einstein equations, lasting $\sim 10^{-36}$ and containing the de Sitter metric. Then we can say that such an locally anisotropic stage was followed by a decelerated homogenizing expansion, first with a radiation dominated era and then by matter dominated era.

The class of off-diagonal inhomogeneous solutions (63) is different for the "diagonal" family of Szeres-Szafron metrics considered in inhomogeneous cosmology. Here we note that the books [28, 29], including references within, provide a comprehensive review of the characteristics, properties and exact and/or inhomogeneous cosmological solutions. From any such diagonal and off-diagonal inhomogeneous metric, we can recover the FRW model, consider solitonic perturbations and analyze contributions of a nontrivial cosmological constant [30].

Our main conclusion is that using generic off-diagonal exact solutions of the Einstein equations, with correspondingly prescribed nonholonomic distributions, we can elaborate cosmological models with exponential expansion and limits to FRW configurations without

additional scalar fields which would be responsible for inflation. In our approach, inflation is modelled by nonlinear off-diagonal interactions and constraints on such a "pure" gravitational dynamics (in [31], we studied a model of anisotropic brane inflation with off-diagonal metrics).

5 Outlook, Discussion and Conclusions

In this work we have provided the essential features on applications in cosmology of the anholonomic deformation method which relate the geometry of nonholonomic manifolds to exact solutions in gravity. In particular, given any physically important cosmological solution of Einstein equations (defining respectively the FRW, Bianchi, Kasner, Gödel or other universes), we constructed new classes of generic off-diagonal exact cosmological solutions which are, in general, locally anisotropic and/or inhomogeneous. Alternately, the nonholonomic deformations described here for cosmological solutions can be viewed as examples of the geometry of nonholonomic distributions and generalized transforms of geometric structures for classical and quantum (non) commutative spacetimes. While we have considered the approach for Einstein's gravity, it is clear that the general constructions can be extended to higher dimensions, various Lagrange-Finsler and string/brane gravity models (albeit with increasing computational complexity in computing off-diagonal higher dimension terms and/or higher order nonholonomic constraints).

Our approach allows us to construct general cosmological solutions in various gravity theories with metrics $g_{\alpha\beta}(u^\tau)$ parametrized in the form

$$
q^2 \times
\begin{vmatrix}
g_1 + \omega^2(w_1^2 h_3 + \omega^2(n_1^2 h_4)) & \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & \omega^2\, w_1 h_3 & \omega^2\, n_1 h_4 \\
\omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & g_2 + \omega^2(w_2^2 h_3 + n_2^2 h_4) & \omega^2\, w_2 h_3 & \omega^2\, n_2 h_4 \\
\omega^2\, w_1 h_3 & \omega^2\, w_2 h_3 & h_3 & 0 \\
\omega^2\, n_1 h_4 & \omega^2\, n_2 h_4 & 0 & h_4
\end{vmatrix},
$$

where the local coordinates are of type $u^\tau = (x^i, y^a)$, for $x^i = (x^3, x^2)$ and $y^a = (y^3 = t, y^4 = y)$ and spacetime signature $(+, +, -, +)$. The coefficients $g_k(x^i)$, $h_a(x^i, t)$, $w_k(x^i, t)$, $n_k(x^i, t)$, $q(x^i, t)$ and $\omega(x^i, t, y)$ can be defined in explicit form (following well defined and quite simple procedures) by integrating and/or differentiating some generating functions. Such metrics depend on certain classes of integration functions and constants in order to define very general classes of exact solutions in Einstein gravity and generalizations. As a matter of principle, any solution of gravitational field equations with certain general matter fields sources can be represented in the above generic off-diagonal form by corresponding frame and coordinate transforms [17]. We have to involve certain additional physical considerations, suppositions on symmetry of interactions and boundary conditions in order to model certain realistic cosmological models and scenaria.

An important issue which we have briefly discussed in this work concerns the most general classes of cosmological solutions with "nonohlonomic" time like variable $y^3 = t$. The priority of the anholonomic deformation method is that we can determine almost all possible types of cosmological metrics and deformations of connection in general form not making approximations with any terms in the associated systems of partial differential equations. The surprising propriety of the introduces nonholonomic deformations of fundamental geometric and physical objects is that the constructions are such way performed that we get separations of the gravitational field equations (with respect to certain adapted frames of reference) which allows us to generate exact solutions. Further approximations (for instance,

for generic off-diagonal metrics depending only on time variables t and with certain pre-scribed spacetime symmetries) are possible, but in such case there are not lost important types of nonlinear interactions/evolutions which can be "lost" during approximations for deriving effective (simplified) systems of equations.

Furthermore, we have considered various generalizations of the known important cos-mological solutions which may lead to interesting new insights into modern cosmology with locally anisotropic and inhomogeneous physical scenaria and nonlinear interactions. These provide an interesting arena for further exploration in gravity theories and cosmology. In principle it is possible to model various nonstandard inflation, dark energy and dark matter effects not introducing additional/exotic scalar and other fields but imposing certain non-holonomic constraints on the off-diagonal dynamics of gravitational interactions in standard Einstein gravity. In addition, it is clear that there are many interesting directions that can be studied within the framework of the geometry of nonholonomic distributions/frames and off-diagonal metrics in gravity and cosmology. Such investigations became possible after a general geometric method of constructing exact solutions in gravity was elaborated.

References

1. WMAP, Wilkinson Microwave Anisotropy Probe. http://map.gsfc.nasa.gov
2. COBE: Cosmic Background Explorer. http://lambda.gsfc.nasa.gov/product/cobe/
3. Planck Science Team Home. http://www.rssd.esa.int/index.php?project=Planck
4. Nojiri, S., Odintsov, S.D.: Non-singular modified gravity: the unification of the inflation, dark energy and dark matter. AIP Conf. Proc. **1241**, 1094–1099 (2010). arXiv:0910.1464
5. Sotiro, T., Faraoni, V.: $f(R)$ theories of gravity. Rev. Mod. Phys. **82**, 451–497 (2010)
6. Capozziello, S., Francavaglia, M.: Extended theories of gravity and their cosmological and astrophysical applications. Gen. Relativ. Gravit. **40**, 357–420 (2008)
7. Calogero, S., Heinzle, J.M.: Bring cosmologies with anisotropic matter: locally rationally symmetric models. arXiv:0911.0667
8. Sung, R., Coles, P.: Polarized spots in anisotropic open universes. Class. Quantum Gravity **26**, 172001 (2009)
9. Pontzen, A.: Rogue's gallery: the full freedom of the Bianchi CMB anomalies. Phys. Rev. D **79**, 103518 (2009)
10. Frolov, A.V.: Kasner-AdS spacetime and anisotropic brane-world cosmology. Phys. Lett. B **514**, 213–216 (2001)
11. Roberts, M.D.: The Kasner brane. gr-qc/0510005
12. Erickson, J.K., Wesley, D.H., Steinhardt, P.J., Turok, N.: Kasner and mixmaster behaviour in universes with equations of state $w \geq 1$. Phys. Rev. D **69**, 063514 (2004)
13. Gödel, K.: An example of a new type of cosmological solutions of Einstein's field equations of gravitation. Rev. Mod. Phys. **21**, 447–450 (1949)
14. Obukhov, Yu.N.: On physical foundations and observational effects of cosmic rotation. In: Scherfner, M., Chrobok, T., Shefaat, M. (eds.) Colloquium on Cosmic Rotation, pp. 23–96. Wissenschaft und Technik Verlag, Berlin (2000). arXiv:astro-ph/0008106
15. Chrobok, T., Obukhov, Yu.N., Scherfner, M.: On closed rotating worlds. Phys. Rev. D **63**, 104014 (2001)
16. Barrow, J.D., Tsagas, C.G.: The Gödel brane. Phys. Rev. D **69**, 064007 (2004)
17. Vacaru, S.: On general solutions in Einstein and high dimensional gravity. Int. J. Theor. Phys. **49**, 884–913 (2010)
18. Vacaru, S.: Finsler black holes induced by noncommutative anholonomic distributions in Einstein gravity. Class. Quantum Gravity **27**, 105003 (2010)
19. Vacaru, S.: Exact solutions with noncommutative symmetries in Einstein and gauge gravity. J. Math. Phys. **46**, 042503 (2005)
20. Vacaru, S.: Parametric nonholonomic frame transforms and exact solutions in gravity. Int. J. Geom. Methods Mod. Phys. (IJGMMP) **4**, 1285–1334 (2007)

21. Vacaru, S., Stavrinos, P., Gaburov, E., Gonţa, D.: Clifford and Riemann-Finsler structures in geometric mechanics and gravity. In: Selected Works, Differential Geometry—Dynamical Systems. Geometry Balkan Press, Bucharest (2006). www.mathem.pub.ro/dgds/mono/va-t.pdf and arXiv:gr-qc/0508023
22. Vacaru, S.: Finsler and Lagrange geometries in Einstein and string gravity. Int. J. Geom. Methods Mod. Phys. (IJGMMP) **5**, 473–511 (2008)
23. Miron, R., Anastasiei, M.: The Geometry of Lagrange Spaces: Theory and Applications. FTPH, vol. 59. Kluwer Academic, Dordrecht, Boston, London (1994)
24. Mukhanov, V.: Physical Foundations of Cosmology. Cambridge University Press, Cambridge (2005)
25. Weinberg, S.: Cosmology. Oxford University Press, London (2008)
26. Grishchuk, L.P., Doroshkevich, A.G., Novikov, I.D.: The model of "mixmaster universe" with arbitrarily moving matter. Z. Eksp. Teor. Fiz. **55**, 2281–2290 (1968) (in Russian). Translated in: Sov. Phys. JETP **28**, 1210–1215 (1969)
27. Ellis, G.F.R., MacCallum, M.A.H.: A class of homogeneous cosmological models. Commun. Math. Phys. **12**, 108–141 (1969)
28. Kramer, D., Stephani, H., Herdlt, E., MacCallum, M.A.H.: Exact Solutions of Einstein's Field Equations, 2nd edn. Cambridge University Press, Cambridge (2003)
29. Krasinski, A.: Inhomogeneous Cosmological Models. Cambridge University Press, Cambridge (1997)
30. Kraniotis, G.V., Whiterhouse, S.B.: General relativity, the cosmological constant and modular forms. Class. Quantum Gravity **19**, 5073–5100 (2002)
31. Vacaru, S., Gonţa, D.: Off-diagonal metrics and anisotropic brane inflation. arXiv:hep-th/0109114; published in Chap. 9 of [21]